

Liberal Arts Inspired Mathematics

- a report -

How to bring cultural and humanistic aspects
of mathematics to the classroom
as effective teaching and learning tools¹

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1 What is this?

This is a report of an on-going quest for a way of teaching university level mathematics in a humanistic way. The quest has not been very systematic and it has no particular method. Rather it is, and has been, guided by conversations and readings and practical experiments for many years. Furthermore, it is just during the last couple of years that I've understood that it is indeed a quest for a *humanistic mathematics teaching* or perhaps that such a phrase could be an appropriate characterization of it. Happily, I've come to realize that I'm not the first one to think along these lines.

A main part of the project was the journey I made to the United States during seven weeks in the fall term of 2012 to visit Liberal Arts Colleges to study and discuss mathematics teaching. Writing up a report of that trip is the concrete motivation behind the present text. But there are more to discuss.

Though it remains to explain what humanistic mathematics could be - and I hope the present text will convey some of the meaning that could be given to the concept - I think there are at least two cornerstones to an attempt to teach university mathematics humanistically. One of them is a serious discussion with the students about the nature of mathematical objects. Another one is an acute awareness by both students and teachers that what is studied is a language. I will return to these topics towards the end of this text in sections 6 and 7.

Sections 2 through 4 tries to paint the context and the background to the project. The 2012 College trip is reported in section 5. An ongoing report of the project can be found at my weblog **Mathematics as a Humanism**¹.

I should get started, but just two more things. First, this text is written in a personal way that I think is appropriate for a project like this and it mirrors the fundamental humanistic nature of the project. What conclusions I arrive at, I cannot support with solid empirical data, but I hope they will be worth considering anyway. Much is anecdotal and impressionistic. Still it is an academic text although the conventions of academic writing is not strictly adhered to. There is no way that I can refer to everything that has been written on this subject because even though I have the feeling that I've read a lot, it is surely just a little of everything that has been written. My reading has been mostly unsystematic and even leisurely, and I often find myself in the awkward position of not knowing from where I've picked up ideas. Furthermore, as will become clear, I don't have many answers to offer, I will rather pose questions that need further investigations. It is, in fact, a quest².

2 A Paradoxical Situation

A little bit of reflection based on a - not too superficial - look at human history and on our present day society, make it apparent that mathematics is at the center of it all. There are even theoretical physicists that play with the idea that the very bedrock of reality is mathematics, but we need not go as far out in speculation as that [1]. It is enough to realize that mathematics is one of our languages, not easy to learn, but once learned it is the same for everyone,

¹<http://libartinspmath.wordpress.com/>

²According to Merriam-Webster's 11:th collegiate dictionary: "a chivalrous enterprise in medieval romance usually involving an adventurous journey".

except for the natural language that we have to wrap around it. This fact alone, the need for a *meta-language*, tells us that mathematics is part of the human culture. If no meta-language would have been needed, if it had sufficed with just the symbolic syntax and semantics of formal mathematics, then we would have been machines, not human beings.

But there is a visibility problem. Few people use more than the most trivial mathematics in their daily life or at their workplace. The applications of mathematics are invisible [2]. What is not invisible, however, is school mathematics. For most people, learning mathematics is a struggle. In school, mathematics and its esoteric language, is highly visible. Except for the few who have been bequeathed with a natural talent for mathematics. For them, if they continue to become mathematicians or mathematics teachers, mathematics becomes a natural language that they are for the most time unaware of. As teachers they may be unaware of the fact that they speak and write a language foreign to their students. I get the impression that mathematics is often taught as if it were mere syntax.

I think this is at the center of the difficulties with mathematics teaching and learning. This leads up to one of the main points of this report: *The Language Teaching Metaphor* of which I will write more in section 6. This is also strongly connected to the second major issue: *The Nature of Mathematical Objects* as discussed in section 7. These issues are connected through the question: What is the language of mathematics trying to say anything about?

3 Introduction and Intention

What's new - if anything?

Anyone interested in the kinds of questions I raise here will see that my thinking is not very original. Much, if not all, of what I write has been written before.³

In Sweden there has been an ongoing conversation - though not very widely known as far as I understand - for a couple of decades about cultural aspects of mathematics. A publication that opens up into this discourse is *Det matematiska kulturarvet* [3]. Another route is the Ph.D dissertation on "Mathematics and Bildung" [4]. Swedish mathematicians who have written about language aspects of mathematics are Lennerstad [5] and Kiselman [6]⁴.

What is perhaps new here is my focus on teaching and learning. Philosophy of mathematics and humanistic aspects of mathematics are very interesting subjects in themselves, but my main focus is how they can be made the rock bottom foundation of designing courses and teaching methods. I've seen very little written about that.

³I will cite other authors when I have a specific reference, but mostly I don't have that. I know I have picked up from many sources, but I do believe that correct referencing in a subject like this practically impossible. Most likely there are many thinkers who have thought parallel thoughts more or less independently of each other and most likely have been influenced in the same way as I've been influenced.

⁴All these references are written in Swedish.

Constraints and opportunities

There are some assumptions that delimits my project. I'm not teaching mathematics majors. We don't have that in Sweden although a corresponding set of students could be those who go to University to study mathematics or to a mathematics heavy educational program, like engineering physics. So my focus is not on students with particular talent and interest in mathematics. My focus is on students with no particular mathematics talent or interest. Some of them might even detest mathematics or just feel queasy about the subject. They have had mathematics for 10 to 12 years in school but their knowledge is weak.⁵ Or perhaps I should phrase it like this: they do have a lot of implicit and disconnected knowledge in mathematics, but it doesn't make sense to them. It's like pieces of a jig-saw puzzle. These pieces need to be scattered on big table, all turned over with the right side up, then be put together into a coherent whole, providing context and adding in lost or never found pieces. This is an opportunity.

I'm also not discussing pre-college or pre-university mathematics teaching. I'm primarily interested in teaching and learning mathematics at the college and university level. This is where I'm active. Important as earlier stages in the mathematics education are, that's nothing I can do anything about. My objective is to try to enhance learning for students I actually meet.

In Sweden this corresponds to ages 19+, that is, young adults and adults. In the US students start college at around 18 years of age. So Liberal Arts experiences in mathematics teaching are highly relevant for the Swedish situation.

My basic assumption is that, precisely because the students are adults, we can be explicit about teaching and learning methods, we can use *reason* on a meta-level, so to speak. We can discuss teaching and learning explicitly with the students in a way that may be impossible, difficult or not appropriate for younger students or children.

4 Background

Let me sketch the background to this project and in particular the Liberal Arts College trip. The first concrete idea came in the early fall of 2010 when I sat down to write a contribution to a Swedish anthology [7] about Liberal Arts education. The idea behind that was to discuss various aspects of the liberal arts tradition and how it could inspire Swedish higher education. All the authors had some personal experience from the American tradition, many from having spent a term at a college under the STINT⁶ *Excellence in Teaching Scholarship*⁷. I volunteered to write about mathematics.

When I started to write, I realized I knew next to nothing about mathematics teaching at Liberal Arts colleges. During the writing process I learned

⁵If any student is reading this, please don't be offended. I'm not saying you are dumb. I know myself what it means to have problems with mathematics. In seventh and eight grade in school, I completely lost track of what was going on in the mathematics classes. I understood nothing. But I came back. I came back because I knew I wasn't dumb and since I had liked mathematics I retraced a few years and started over again on my own. It was then that I really started to learn the language of mathematics.

⁶The Swedish Foundation for International Cooperation in Research and Higher Education

⁷I was at Skidmore College in Saratoga Springs, NY, during the fall term of 2004.

about the corresponding Swedish discussion (referred to above) on mathematics and "bildung" from the last decade, that I had missed. It became a learning experience.

So why was I at all interested in going in this direction? It goes back to a long time interest in the history and philosophy of mathematics. Also there is my interest in didactics of physics and mathematics. I worked for six years at a gymnasium (corresponding to school years 10-12). I then came into contact with learning theories based on meta-cognition, constructivism and Ference Marton's research on deep and surface learning strategies [8].

About ten years ago, a year or so after I started to work at the University of Borås, I planned and carried through a pedagogical experiment in mathematics together with two colleagues [9]. The background to that project was discussions among mathematics teachers at the institute about the weak background knowledge in mathematics among the incoming students. I wanted to do something about it. The experiment did not go very well. Some of the ideas were good and I still believe in them, but our implementation was too weak and I now realize that we missed many important ideas that I just recently have become aware of. After that, I got the opportunity to study computer science for some years, and my teaching also turned to programming courses as well as basic courses in natural science (for students lacking that from high school).⁸ But the last couple of years I've moved back to mathematics teaching. And I see that not very much has changed, at least not to the better. Incoming students are still very weak in mathematics.

There is one special circumstance that is of importance. In Sweden we have educational programs in engineering that are 3 years of duration. There are just two mathematics courses in general (linear algebra and calculus), sometimes a third course or a course in mathematical statistics. The standard length of a Swedish university course is seven weeks. That means that 3-year engineering students have one half semester of mathematics (they take two courses in parallel), which is not that much. So the resource in terms of available time is scarce. It is of course almost impossible to go very deep in calculus in such a short time. Most engineering students have had some calculus in the gymnasium, at least a brief encounter with derivatives, but some haven't seen integrals. So the problem would seem to be unsolvable.⁹

Why then try a humanistic approach to engineering mathematics? The students that enter engineering programs have 10-12 year of mathematics from school. That is indeed a lot, and they do have a lot of knowledge, but it's a kind of implicit knowledge. It's not really workable knowledge, very many have problems with simple numerics and algebra. Functions are dim concepts. As I wrote above, it's like all the acquired knowledge from 12 years of school need to be scattered on a large table, like jig-saw puzzle pieces, and then put together into a coherent whole, with new pieces added and context provided. That sounds like a humanistic endeavor.

⁸Studying computer science, in particular the theory of programming languages, turned out to be a valuable experience. Programming languages share many features with mathematics, and I came into contact with computer scientist's explicit thinking in terms of *syntax*, *semantics* and *pragmatics* of programming languages. It has gradually dawned on me that this is a fruitful way of thinking of mathematics too. More on that below in section 6.

⁹The gymnasium courses in mathematics have just been changed again, hopefully to the better, and the first students having had the new courses will arrive at the university in 2014. It remains to see what difference it makes.

Even though mathematics is a supporting topic, subordinated to technology, that does not mean it must be taught and learned that way. The question is: how can the scarce resource in terms of time be used in an effective way? The students must be engaged so that they are prepared to invest extra study time for home work.

Rationale for Liberal Arts¹⁰

In Europe, in connection to the Bologna process, there is a focus on "employability" as the overall outcome of education. University education in Sweden is almost exclusively vocational. Of course it cannot be denied that most people study in order to get a good and interesting job and a decent career. There's nothing strange or wrong about that.

At the same time, society becomes ever more complex, and humanity faces outstanding challenges. We know nothing about the future. Education should also prepare for that. Then there are democratic, humanistic and personal values connected to education that cannot be reduced to a career in the workplace. I'm very sympathetic towards the kind of generalist knowledge liberal arts education fosters. Even from a purely pragmatic point of view, the cornerstones of a liberal education are good both for the personal career and for the bettering of society. Studying mathematics from that point of view should make sense.

Humanistic mathematics

When I first put pen to paper and began to write about this project, early last year, I was worried that the idea of mathematics as a humanism would sound strange. I had arrived upon the idea in connection with a book-writing project with a colleague, but curiously enough, during all that time the very phrase "humanistic mathematics" never crossed my lines. So I wasn't aware of the fact that the term *humanistic mathematics* was fairly well established in the US. It actually came as a pleasant surprise, although thinking about it, it would have been really strange if no-one had thought along these lines before. Anyway, such was the extent of my ignorance just a year ago.

To view mathematics as part of the human culture, is of course not new. I think most people with an interest in mathematics, sooner or later, come across the humanistic aspects of mathematics.

I asked myself when I first came into contact with this view of mathematics. I thought about reading Davis' and Hersch's *The Mathematical Experience*. I found this book in Camden Market in London in 1986 when I worked at Queen Mary College as a post-doc research fellow. We used to go there on Sundays as it was within convenient walking distance from Kentishtown where we lived.

The question struck me while reading the PhD dissertation "Matematik och bildning" by Lars Mouwitz, a Swedish mathematics teacher and scholar [4]¹¹. It then occurred to me that my first encounter with cultural aspects of

¹⁰For readers unfamiliar with the American Liberal Arts tradition, here's two references [10, 11] of which the second one discusses how ideas from the Liberal Arts can help improve Swedish higher education.

¹¹The word "Bildning" - which is the same word as the German "Bildung" has no direct translation in English as far as I know. But it has the same connotations as "Liberal Arts".

mathematics must have been in high school in the early seventies when I read all popular science books in physics and mathematics that I could find at the local library. In particular, there was the anthology Σ igma in six volumes which I must have bought sometime because it is still on my bookshelf. When I picked it out a while ago, I got hold of volume 6, and randomly it fell up on an excerpt from Oswald Spengler [12]. My eyes fell on the sentence "The mathematic, then, is an art." This caught my attention and I read the full article which actually turned out to be very interesting, its strange context notwithstanding. Apparently, Spengler wrote a long section on the meaning of numbers in his "The demise of the West".¹²

Σ igma was published in the US in 1956 under the title "The World of Mathematics" and it is a proof that awareness of cultural aspects of mathematics goes at least that far back. The editor, James R. Newman had been working on the project since 1944. Strangely enough, some time ago, I found the English version in an antiquarian book shop nearby. It was in good condition, almost unread, and there was a bookmark in an article by Bertrand Russel "Mathematics and the metaphysicians".

A personal anecdote

Modern mathematics would be impossible without a symbolic language, without *signs*. Anyone with a talent and interest in mathematics picks up this language in a more or less painful process. Allow me a personal anecdote. When I went to school, it could have been in fourth grade, I found a mathematics book in the bookshelf at home. It was a 600 pages textbook for business mathematics containing elementary arithmetic, algebra, logarithms, practical geometry and trigonometry [13]. It was my father's book. I was intrigued by the chapter on algebra "figuring with letters". Right on the first page there was an expression

$$a + a + b + b$$

that mystified me. I don't know how many hours I spent contemplating it. The memory is clear and when I now, many years later, look up the page again, I remember (somewhat romantically) how I rushed home from school and sat at the kitchen table with a sandwich and glass of milk, trying to understand what it could possibly mean. The explanations in the book are not bad, they are actually quite good (compared to many modern books), but I still couldn't understand how you could "add" letters. If you try to add a to b , what do you get? You get nothing! The plus-sign asks you to add numbers, and I could do that. But to be asked to add a and b was, ... well what was it? At that time I couldn't put words to what was wrong with the request. Today I would call it a "category mistake". Letters are for writing words, not for adding. When I tell this story to colleagues they do not seem to recollect any similar problems. To me the process of understanding the symbolic language of mathematics was, not painful really, but an intellectual challenge, or perhaps I was just to young.

In Swedish, "bildning" is something you could possess. Perhaps the words "erudition" and "wisdom" convey part of the meaning. Could one say that whereas "bildung" is an object, "liberal arts" is more of a process?

¹²One of his quite intriguing ideas is that there are different mentalities in mathematics. He writes that the analytic geometry of Descartes (and Fermat) is conceptually different from the ancient Greek geometry. People "saw" different things.

How the College Odyssey came about

I enjoy traveling in the US. Apart from short conference trips, me and my son Erik made a coast-to-coast trip in 1997 by train and car. And then I spent four months in upstate New York in the fall of 2004. So I had had in the back of my mind for some time, the idea of the next trip. When I started to write about mathematics in the anthology about Liberal Education, it occurred to me that I could actually go back to the US and study mathematics teaching in particular. This was in the fall of 2010. I wrote a letter to Sheldon Rothblatt who encouraged me and directed me to Lynn Steen to who I also wrote. From professor Steen I got more encouragement as well as a couple of reading tips, one of them his article about the invisibility of mathematics [2].

There the matter rested for a year. I didn't know how to proceed or how to finance the project. Eventually I got the idea not to worry about funds but instead just write to colleges and present the idea. And anyway, without concrete contacts in the US, nothing would come of it anyway. So for a month or two in the early fall of 2011, I struggled with a letter (an e-mail actually) to Liberal Arts colleges. Then one night in October, I systematically ran down the list of top ranked Liberal Arts colleges and looked up e-mail addresses to heads of mathematics departments. A few more weeks of procrastination went by, before I summoned the courage to actually send off the e-mails. About half of the colleges answered and the answers were all very kind, some of them enthusiastic about the idea. Having so gained confidence that the project made sense, I got the very natural idea to simply ask my own institute to finance the project. It did. Then I applied and got additional funding from a Swedish foundation¹³. Suddenly it was all clear that the College trip was going to be.

5 Notes from the Colleges

I made the college trip to learn about mathematics teaching at Liberal Arts colleges. I must have had some preconceptions about it, but they were not very explicit or conscious. I knew from my visit to Skidmore College in 2004, to the extent that I thought about it, that the teaching in general in some sense could be described as "conventional" but of high quality. I had sat in on quite a few classes in all sorts of subjects.¹⁴

If there had been dramatic differences in the teaching to what I was used to, then certainly I would have noticed. But since my focus in 2004 was not on mathematics in particular, I had the feeling that there must be something more to learn. What are they doing and how? But I needed to formulate a more concrete question. You can't do research without a question. Gradually, the question, the key question as I called it, came out as

In what ways do the Liberal Arts environment influence the way
mathematics is taught as compared to other colleges?

But my thinking wasn't very clear, I had somehow conflated humanistic mathematics with mathematics as taught at Liberal Arts colleges. I embarked on the

¹³Stiftelsen Långmanska kulturfonden.

¹⁴I in particular enjoyed Tad Kuroda's classes on Colonial America.

trip thinking that I would find examples of humanistic mathematics teaching at Liberal Arts colleges. That was not really to be.

In the following sections I will briefly, and unsystematically, review some of my conversations at the colleges. As I had expected, all the visits turned out to be different, and this is reflected in the reports below. Looking back at the conversations, I cannot really say that they revealed anything dramatic that I had never thought about before, rather they deepened my understanding of mathematics teaching and underlined and put into new perspectives things that I had observed, read about and thought. I gained confidence that my thinking wasn't completely crazy. And a few things were definitely new.

Some themes ran through all the visits. To just chronologically review them would not make any sense to the reader. Instead I have tried to put some logic to the discussions and written about them under the heading of a college where, so to speak, my understanding began to jell. However, I have to begin with a general comment.

Historical and Philosophical Backdrop

The United States, just as Sweden and many other western countries, are not doing well in international tests in mathematics. There is a history of declining standards and watered-down content. In Sweden this is part of the folk-lore among university teachers. In part, possibly a to a large part, this is due to higher education being transformed from concerning only a small elite to include at least half of the population. This is of course a good development going in parallel with democratic and egalitarian ideals, but it has created problems for teaching and learning. The pedagogy of a hundreds years ago is not likely to be suitable in modern societies. Then, most students were likely to sustain doubt and just carry on, and we don't know how much they actually learned or understood. Modern students need motivation, they voice discontent and we as educators are indeed aware of how much (or little) they learn or understand.

There is a tendency among educators always to lament the present situation: things are bad and they are becoming worse, something must be done! Sometimes the quality of the students and their knowledge is lamented, sometimes the teachers, the administrators, the teaching schools and the politicians, often all of them. An almost caricatural illustration, in particular regarding the university and college teachers, of that is Morris Kline's book *Why the Professor Can't Teach* [14]. If his picture of American mathematics teaching in 1977 is correct, then surely the situation must be much better today.¹⁵ Things are changing, sometimes to the better sometimes to the worse. My outlook is that "things" are "generally improving" over time. That does not mean that there aren't any problems, quite to the contrary, there are always problems but as we try to fix them, things are improving. Unfortunately, solutions to old problems tend to create new problems. The opposite, and quite common, view: "things" are "generally getting worse", is not tenable. If that was true, then looking back in history, things would generally always be better before. Eventually we end up back in the caves [15].¹⁶

¹⁵The book can, and perhaps should, be read as written "tongue in cheek" but he must have felt he had an urgent message to get across.

¹⁶A simplification of K. Popper's philosophy of science (and politics) is: the point is not to do things right, it is to see the problems and try to fix them. Then we get progress.

5.1 Beloit

Beloit College was my first college visit and it turned out to be a very good start. I drove up from Clinton, Iowa, a nice Sunday morning in early September. I had spent a few days in Iowa - a state I long had wanted to see - after dropping down in Minneapolis. Now I crossed the Mississippi on an old rusty iron bridge, but I didn't see much of the great river. In Rockford I phoned Paul Campbell who was my contact at the college. We met outside the college guest-house and then went to a heritage day nearby, and later for dinner at his house.

When Paul Campbell answered my letter in March, he also directed my attention the Journal of Humanistic Mathematics as well as attached an article he had written about calculus [16] and an answer to it [17].

At Beloit I immediately hit on several themes that were to resurface throughout my trip:

- Calculus
- The "Is Algebra Necessary" discussion
- Mathematics as a liberal art
- Mathematics as a humanism

Let me start with the last two items.

Mathematics as a Liberal Art

I had a short discussion with David Ellis on the Friday just before the seminar talk I gave at Beloit. Since time was short I simply tried out the key question in whatever early phrasing it had at that time. The answer was short and succinct:

Mathematics **is** a liberal arts subject.

Perhaps it is so that *liberal arts* are often equated with *the humanities* or with *humanism* and that mathematics is not naturally sorted in under these categories. Most often mathematics is thought of as standing closer to the natural sciences and technology because of the spectacular success of applied mathematics.

Paul Campbell had looked up the words *humanity* and *humanism* in a dictionary and found

humanities:

- (a) The languages and literatures of ancient Greece and Rome; the classics.
- (b) Those branches of knowledge, such as philosophy, literature, and art, that are concerned with human thought and culture; the liberal arts.

humanism:

- (a) A system of thought that rejects religious beliefs and centers on humans and their values, capacities, and worth.
- (b) Concern with the interests, needs, and welfare of humans.

In Swedish we don't have this distinction: the Swedish word corresponding to **humanity** has the meaning of the human species. This means that **humanism** is used both for a philosophy and a world view and for a wide-ranging set of academic disciplines. I now realize that this ambiguity must have been behind my feeling (when planning the project) that the phrase "mathematics as a humanism" might sound strange.¹⁷ I now believe that thinking of mathematics a humanism can thrive on this ambiguity.

However, there should be no doubt that mathematics is indeed a liberal art. In the old classification of the seven liberal arts, the first three were called the Trivium and the consisted of Grammar, Logic and Rhetoric. These were the "language" liberal arts. Logic has been a mathematical subject ever since George Boole, perhaps even since Leibniz, and certainly since Frege and Russell.¹⁸ Grammar, widely interpreted not just as grammar of natural languages, also includes the formal grammars of programming languages and is by now a mathematical science. The next four liberal arts, the Quadrivium, consisted of Arithmetic, Geometry, Astronomy and Music. The first two have always been mathematics. Astronomy was mathematics at the time, but have since moved into the natural sciences. Music, at the time of Pythagoras, was mathematical music - the harmony of the spheres.

With some anachronism, all of the classical liberal arts except Rhetoric, was mathematical. Mathematics could easily be thought of not just as a liberal art, but **the** liberal art.

So the answer that Dave Ellis gave: that mathematics is a natural subject at a liberal arts colleges is a good answer, but it still begs the question. Really, in what way is this fact visible in the teaching at liberal arts colleges today? My quest had just started.

5.2 Carleton and Macalester

I visited Carleton College and Macalester College during the same second week in September. I stayed in Northfield, the home of Carleton and St. Olaf College. I did not visit St. Olaf but met with people from there. In contrast to the quite leisurely week in Beloit, the week in Northfield was much more intense. I had spent the weekend in Chicago, visiting my old friend Jean Capellos, who lived in the flat above ours in Kentishtown, London in the 80's. On the way back to Northfield I visited a railway museum in Illinois. But I lingered to long and it grew dark as I drove north on highway 61. I crossed the Mississippi at La Crosse, but did not see the river this time either.

My contacts in Northfield were Deanne Haunsperger from Carleton and Karen Saxe from Macalester. They had me scheduled from early morning till late night. The classes I went to and the people I spoke to are listed in the acknowledgments. At Macalester I was also invited to an outdoors lunch with the faculty at the college president's house. At Carleton I sat in on a morning of mathematics faculty retreat.

¹⁷Clearly, "humanism" as defined above was what Jean-Paul Sartre had in mind when he gave the talk "L'existentialisme est un humanisme".

¹⁸For a popular account, see [18].

Calculus, Sputnik Calculus, Reform Calculus and Calculus

Already after three colleges it became clear to me that calculus was at center stage. I had discussed the Calculus Reform with Paul Campbell and Bruce Atwood at Beloit and now the subject cropped up again. I remembered reading about it in the *Mathematical Intelligencer* in the 1990's, but I hadn't thought about calculus and calculus courses in particular during the planning of the project.¹⁹

Now I saw calculus as an American preoccupation. Of course, the importance of calculus is obvious: it was the second major breakthrough (after analytic geometry) of Western mathematics in the 17:th century after centuries of poring over the old surviving manuscripts of the classics. The calculus of Newton and Leibniz then rapidly developed over the next two hundred years together with mechanics, hydrodynamics, thermodynamics and electrodynamics. It became the language of natural science and technology. And it spurred an enormous evolution in mathematics itself.

However, this was primarily an European development. American mathematics lagged far behind, both in teaching and research, up to the early twentieth century [2]. It was a country of mainly poor immigrants trying to build a civilization in the wilderness. Still, schools were set up from the very beginnings and the first colleges were soon to follow. But they catered to the needs of a pragmatic pioneer society.

By 1900, calculus was a standard college subject [14], but few high-school students took algebra or any higher mathematics [19]. Then came the world wars, in particular WW-II, and the rise of the US to a world power and the influx of European mathematicians and scientist fleeing Nazi persecution. The US became a leading power in science.

During lunch after the mathematics faculty retreat at Carleton, the topic of the Sputnik and calculus came up again.²⁰ The shock of the Soviets putting up a satellite²¹ before the US led to a huge increase in government money spent on education, in particular science and mathematics.²² Ever more high-school and college students took algebra and calculus in some form.²³

Not surprisingly, the teaching methods that had worked - well enough at least - when higher mathematics was an elite subject did not work very well when it became a mass education subject. This became increasingly clear in the 1980's. Incidentally, this was also the time when the electronic calculator had been around for some time and its impact started to be felt in mathematics teaching. Perhaps more importantly, research into how human learning actually comes about - as opposed to wishful pedagogical ideologies - became better known.²⁴ Education is always lagging behind.

¹⁹My thoughts were focused on algebra where the weakness of student skills and understanding are already apparent.

²⁰I talked to Andrew Gainer-Dewar and Brian Shea, two young faculty members.

²¹Sputnik, October 4, 1957.

²²National Defense Education Act.

²³At the turn of the millennium, roughly 700 000 students enrolled in college-level calculus courses [20]. More estimates of enrollment in US calculus courses can be found at http://www.maa.org/columns/launchings/launchings_04_07.html.

²⁴Perhaps there is an analogy here to scientific method, where thinkers such as Popper, Lakatos, Feyerabend and Kuhn were more interested in how research is actually done, rather than prescribing how it ought to be done.

The "Calculus Reform Movement", was an initiative by the National Science Foundation, running from 1988 to 1994. It was based on recommendations from a small conference of mathematicians and teachers, the Tulane Conference [21, 22]. The focus was both on content and teaching methods, in particular student-centered learning, project work and writing, as well as use of calculators and computers in mathematics teaching.

Upon reading about the calculus reform, what it tried to accomplish and how it succeed and failed and the criticism towards it, I must say I'm confused. As an outsider there is no way I can do justice to this complicated historical process. The historical background is described in the report [23]. Teaching calculus is difficult in whatever way you try to do it and many students will fail for many different reasons. So if there is one more or less traditional way of teaching, it is easy to see its shortcomings: too much blind calculational drill, not enough understanding of concepts, et cetera. So something else is proposed instead. Let me quote from the preliminary evaluation in [24]:

"Institutions nationwide have implemented programs as part of the calculus reform movement, many of which represent fundamental changes in the content and presentation of the course. For example, more than half of the projects funded by NSF use computer laboratory experiences, discovery learning, or technical writing as a major component of the calculus course, ideas rarely used prior to 1986 [reference removed]. The content of many reform courses focuses on applications of calculus and conceptual understanding as important complements to the computational skills that were the primary element of calculus in the past. It is believed by many that such change is necessary for students who will live and work in an increasingly technical and competitive society."

The reform proposals were controversial and new problems arose where they were adopted. The two sides of the debate can be illustrated by references [25] and [26] which, however, I don't think stand so far apart after all.²⁵ Examples of implementations of the reform efforts are described in [28] and [20]. I get the impression that the long-term effects of the reform movement has been rather small, or perhaps what was working well has been absorbed into mainstream calculus teaching. The main objectives of the movement: focus on conceptual understanding, more varied teaching methods that involves active student engagement, use of modern technology, does not sound at all provocative today [29].

But the story continues to unfold, of course [30]. There are still concerns about calculus and calculus teaching among the community. David Bressoud at Macalester told me about a national survey of Calculus I instruction conducted by the MAA in 2010. One of the goals of the study is to improve calculus instruction across the US. The reader is referred to [31].

In the light of all this, I reread Paul Campbell's article [16] about the problems of calculus teaching once again.²⁶ It does not lend itself to a short summary

²⁵Mumford refers to Lancelot Hogben [27], which I also read in my youth, a piece of very good liberal arts writing in mathematics.

²⁶The follow up discussions are also interesting, see [17, 32, 33].

and I guess it must be read to a backdrop of the calculus reform movement, because as it says "Didn't we go through all this already in the 1990s in the 'calculus reform' movement?" Obviously, the calculus reform did not solve all problems for all time. Which of course is not to be expected. If I understand the article correctly, Campbell's main complaint is on how calculus is taught. Let me quote two key sentences. The first one in relation the calculus reform

"What I offer is a philosophical critique about how we should teach calculus so as to situate it in the mainstream of intellectual pursuits."

The second one just at the beginning of the article

- Either "intellectualize" and "pragmatize" calculus - return calculus to the world of ideas and applications [...];
or else
- acknowledge that calculus is basically a utilitarian skills course, stop giving liberal arts credit for it, [...]

This is then elaborated in the text. I find it intriguing to think about calculus teaching in this way: both intellectualize it and pragmatize it. To intellectualize it is to teach it as humanity, as Campbell does indeed write. But this is in no way contradictory to the practical and applied aspects of the subject. I think this is one of the strong points of thinking about mathematics as a humanistic subject. First: it does move the focus to aspects of mathematics (the cultural, philosophical, historical) that are often forgotten in teaching. Second: it does include all the applications because this aspect is just a part of any human endeavor.

It seems that the calculus reform movement did not do this. In what I have read about it, there is no mention of humanistic aspects of mathematics. There is a curious sentence in the historical background text in the report of the calculus reform movement

"While mathematics was always essential for most scientific disciplines, in 1960 calculus was still viewed by many outside the sciences and engineering as a liberal arts subject."²⁷

Thinking about this, I now see another meaning to humanistic mathematics. Humanism is opposed to any form of extremism or simplification of reality, also in pedagogy. The humanist sees the complexity of reality as a blessing.

5.3 Oberlin

My next college stop was Oberlin College in Oberlin, Ohio. I arrived there on a Sunday afternoon after having visited the Henry Ford Museum in Dearborn outside Detroit. I had traveled through Wisconsin and the Upper Peninsula and over the Mackinaw bridge. My contact was Susan Colley who picked me up at the Oberlin Inn for a dinner at her home. Her husband cooked a wonderful dinner of smoked fish, spinach and cornbread while we chatted over a beer. The discussion turned to the notorious "Is Algebra Necessary" article.

²⁷Page 10 in [23].

Is Algebra Necessary?

This discussion came up several times during my college trip, in particular in Beloit and Oberlin.

It was initiated by a New York Times article by Andrew Hacker [34]. The article itself starts out with a picture of the ordeal that mathematics, and algebra in particular, means for most of American high school and freshmen college students each year. Then it says that

”Nor is it clear that the math we learn in the classroom has any relation to the quantitative reasoning we need on the job.”

This is by now a familiar sentiment. School mathematics is felt to be mostly irrelevant. I find the argument quite difficult to answer. One thing that could be said though, is that engineering students do need quite a lot of classic algebra and calculus. Studying natural science and technology without mathematics doesn’t make any sense. Then it is another thing whether most of the engineering graduates actually use much explicit mathematics in their jobs. Many don’t but some do.

Hacker puts the question

”What of the claim that mathematics sharpens our minds and makes us more intellectually adept as individuals and a citizen body?”

and answers that traditional algebra instruction does not do that. We would hope that it does, but I’m dubious that it does. This connects to something else that I’ve had in my mind without being able to formulate it explicitly until I just recently read an article by Leone Burton [35]. One thing I picked up from that article is the importance of keeping apart *content* from *process*, in this case the content of courses and the process of mathematical thinking. Teaching content does not guarantee that the students pick up mathematical thinking. One of my colleagues²⁸ has been saying this for many years ”it does not matter what we teach them as long as we teach them to think”. The sense of this has gradually dawned on me. Of course we must chose relevant content, but often the focus is too much on the content itself, instead of on the processes of mathematical thought.

But this is a tricky question. We do teach problem solving and we do teach proofs, which are indeed examples of mathematical thinking. But I think that it is too much washed up and dried. We show *the results of problem solving* (solving problems in a linear way without errors, false starts or re-tracings). We prove theorems as if they were straightforward mechanical deductions. The creative, tentative and exploratory aspect of mathematical thought is downplayed.

Much of Hacker’s article is anathema to many mathematics teachers. Myself, I’m not so offended by it. And the article actually ends on a very positive note as it asks for alternatives ...

”The aim would be to treat mathematics as a *liberal art*²⁹, making it as accessible and welcoming as sculpture or ballet. If we rethink how the discipline is conceived, word will get around and math enrollments are bound to rise.”

²⁸Magnus Lundin.

²⁹My emphasis.

It seems that many readers of the article miss this point. The way mathematics is taught today is not optimal.

At Oberlin I had a lunchtime meeting with faculty where I described my project. The next day I had a similar meeting with a set of students, they were all mathematics majors.

5.4 Bryn Mawr

After Oberlin I drove through Ohio and Pennsylvania. I got the impression that the country-side in Ohio was scaled down, more European, as compared to vast fields of Iowa and Minnesota. It was beautiful. In Pennsylvania I crossed the Allegheny mountains and stayed the night in the small town of Shawnee after descending Mount Ararat. It was the cosiest little motel with cold rooms and no wifi, but a nice restaurant across the road with hearty Italian food and German October beer. In the misty morning I left Shawnee as the yellow school buses picked up kids waiting along the road. I continued on the Lincoln highway to Bryn Mawr outside Philadelphia.

My contact at Bryn Mawr was Paul Melvin. I had written to the three colleges Bryn Mawr, Swarthmore and Haverford and gotten positive answers. But as my list of colleges had grown a bit long, I got the idea of suggesting that I give a joint seminar at these three colleges. This was arranged by Paul Melvin and on a Friday afternoon I held the seminar. Joshua Sabloff from Haverford College and Thomas Hunter from Swarthmore College, which I had written to, were there.

What's retained and the transfer problem

The question about what students retain from a mathematical education came up in the discussion after my seminar. Thomas Hunter said that it was an experience; of having learned mathematics. Josh Sabloff quite strongly argued that it must be something more than an experience. A heated but good-humored debate followed with Paul Melvin as a mediator. Partly it was a question of the meaning of words like "experience". I listened. Josh Sabloff made up an example of a medical doctor that ten years after the last mathematics course had to read a scientific article about tests of some new drug. How would that person go about judging the evidence put forward in such an article? His answer was: using knowledge retained from a mathematical education.

Such retained knowledge³⁰ could consist of specific things such as reading tables and diagrams, parsing formulas but perhaps more likely abstracted knowledge such as analytical and logical thinking, discriminating between what's important and not, et cetera.

The discussion reminded me about an article by Underwood Dudley, arguing that algebra (mathematics) teaches us to think [36]. This is something we all want to believe. I said, at Bryn Mawr, that it would be very interesting to have some kind of evidence that such deep - and tacit - knowledge is indeed what results from a good mathematics education. Then digging deeper into the texts that I acquired during the trip I found an answer [37] to Dudley's article that

³⁰I'm filling out the discussion in retrospect with things that were not explicitly said, but was implicit in the trains of thought as I understood it

referred to actual research on transfer. This article, which also contains useful references to work on transfer, concludes that

”There appears to be no research whatsoever that would indicate that the kind of reasoning skills a student is expected to gain from learning algebra would transfer to other domains of thinking or to problem solving or critical thinking in general. The lack of such research evidence does not mean that such transfer does not occur or that algebraic reasoning might not have positive effects on problem solving and critical thinking.”

The point is that there seems to be no research showing transfer of algebraic skills to other domains.³¹

When writing these sections I happened upon the article, mentioned above, by Leone Burton [35] that may be relevant for these questions. If I understand it correctly, it argues that mathematical thinking is not learned by learning mathematical content. Indeed, conventional mathematics teaching is mostly concerned with content, not the process of mathematical thinking. This should be relevant for the transfer discussion, since content is eventually forgotten, but the attitude of a mathematical approach to problem solving may be retained.

Granted that mathematics teaches us quantitative, analytical and logical skills that may be retained - perhaps tacitly - long after the details are forgotten, it can perhaps be contended that it does not really matter what parts of mathematics are studied as long as the teaching is good and the studies are serious, and the teaching and learning is focused on the processes of mathematical thinking. These are clearly very interesting and important issues to understand.

The Transfer Problem and Algebra

The connection between the ”Is algebra necessary” discussion and transfer problem was also made in an article by Lynn Steen [39]. Steen writes that

”[...] what he [Hacker] really says is that it [algebra] is not working in the [American] curriculum.”

Steen argues that algebra does not work as a vehicle to convey usable and transferable mathematical skills to the majority of students, most of who go on to careers that do not use much (or any) explicit mathematics. They would be better served by a more varied mathematics curricula along the lines of ”Quantitative Literacy”. The discussion is further commented upon by David Bressoud in [40] who stresses that if we want transfer to occur, we must teach for transfer.

There is of course no question that students aiming for STEM³² careers need to study content relevant for such careers, but a move of towards teaching process, I do think is needed.

These two articles also refer back to earlier discussions of this topic. I will return to one aspect of it later on, but for now, we have to move on.

³¹Paul Campbell, who was also interested in the transfer problem, gave me an article about a pedagogical experiment that indicated that pattern recognition within a narrow mathematical context seemed to transfer more easily from an abstract formulation to concrete representations than between different concrete representations [38].

³²Acronym for Science, Technology, Engineering and Mathematics.

5.5 Skidmore

After Bryn Mawr, I headed north for the last leg of my trip. I was going to Saratoga Springs. On route, I visited the Daniel Boone Homestead and the incredible Roadside Americana which is a huge model railway - an America in miniature. I also passed the Delaware Water Gap. This was a disappointment as there didn't seem to be any way of getting near enough to see anything of interest. Instead, I continued through the enormous mountain ranges of north-eastern Pennsylvania. When I passed into New York it was late afternoon and as I neared Saratoga it was already dark. The rain was pouring down but I recognized the Northway as if I had just driven on it yesterday. Although it was past nine, I met with Steve Goodwin, and we went for a downtown beer in a not too noisy place - it was Saturday evening.

Liberal Arts colleges have what is generally called "First Year Seminars" or variants thereof. These are special seminar courses taken by the freshman students³³ often as part of a wider "First Year Experience"³⁴. The F.Y.S can be anything really, and many colleges offer seminars that are mathematical. A F.Y.S in mathematics should be an excellent opportunity to do something different in mathematics, and it seems that this is what is often done. Most of the students are not planning to study any more mathematics - apart from perhaps the distribution requirement. From an article by Susan Colley at Oberlin College I learned about one way of conducting a first year seminar in mathematics [41].³⁵

This year at Skidmore, the mathematics F.Y.S was conducted by Mark Huibregste who I knew from my 2004 stay at the College. The seminar was focused on Geometry. The students read Euclid, at the moment the first book. There is an on-line edition with interactive pictures that was used in the seminar. To Swedish ears this must sound incredible. Too many it may also sound as a complete waste of time. I mean, Euclid, isn't that 2500 years old stuff? And I admit, I sat down a little prejudiced because I've never really liked geometry myself, in particular not that type of "synthetic" geometry.

But amazingly it worked. Even I understood the proofs and the general drift of the argument: from working with parallelograms and proving theorems on areas over the 5:th (parallel) postulate to an elegant proof of the Pythagorean theorem. And the humanistic aspects of mathematics are there too, as Mark pointed out. It's there in the comment that the Greeks did not have the real numbers and so no quantitative measure of area, it's there in the comment that some areas (lengths really) could not be sized up by the only numbers the Greek knew about (the natural numbers), it's there in one aspect of mathematics that is often so hard to communicate: the logical character of the subject.

From this class I went away with the happy realization that studying Euclid is not at all useless. Here the strength of the liberal arts context shone through.

³³Freshman = 1st y, Sophomore = 2nd y, Junior = 3rd y and Senior = 4th y.

³⁴What a First Year Experience is, is well-known in America, but for Swedish readers here's a short description: Most Liberal Arts colleges have what they call First Year Experience (F.Y.E) for the new students. Apart from a general introduction to college life and studies and a socialization experience, it is also aimed at introducing the students to the particular characteristics of a liberal education. Various seminars are offered centered on various subjects, among them mathematics. The seminars are often inter-disciplinary.

³⁵More examples can be found on the Internet.

5.6 Bennington

During the week in Saratoga I drove the short distance into Vermont to visit Bennington College, very beautifully situated in the mountains outside the small town Bennington. The autumn colors were in full swing and it was a bit chilly in the air. I had use for the pullover I had bought in Saratoga the other day.

My contact at Bennington College was Andrew McIntyre. The college is focused on the humanities and it was the first to include visual and performing arts in a liberal arts education. Mathematics is not a big subject at the college and mathematics and science form a faculty together. As Andrew had told me in e-mail conversations, he had spent time re-thinking the mathematics curricula and syllabi. Reading the course descriptions, they are distinctly different from most other descriptions I've seen. This made it interesting to me. It was Mark Huibregtse who had, in mail conversations in the spring, directed my attention to the mathematics curriculum of Bennington College.

The teaching method was also different. It was much more student-focused, more bottom-up, working from examples and exercises (which the students worked on themselves in groups and individually) towards the general theory. But in a guided way, guided by lists of things to do which I thought of as road-maps into the subject. I also talked to a student who was an Astronomy major. She had been studying at UMass-Amherst, so she could compare the method to the teaching at a large research university. Her comments gave me additional insights into the teaching at Bennington. I thought of the approach as "pedestrian". I had actually tried something like this last spring in a multi-variable calculus course where I worked from examples towards the general theory using matlab for visualization. It worked better than a traditional "theory \rightarrow examples" approach. What was lacking (in my class) was supportive course material. I did it on the run out of necessity, because standing in front of the students the third time I gave the course, I realized I had to communicate. Remember, the students I have are not the equivalent to mathematics majors. Their knowledge of single-variable calculus, even algebra, is weak.

5.7 Colby

I left Saratoga on a Saturday morning. It was the Columbus weekend with Monday a holiday (although not a College holiday) and I had planned to use three days for the trip up to Maine and Colby College and Bates College. It was leaf-peeking season. I had read about crowded roads, but saw nothing of that except the road leading up into the White mountains. I turned around and skipped that part. I did however have some problems finding a room for one of the nights.

At Colby I had the most intense afternoon on the whole trip. During four hours I talked to eight faculty members. But before that I had lunch with Scott Taylor. I learned that the department offers a course "Mathematics as a Liberal Art" for students who need to fulfill the distribution requirement. It is in general not taken by the mathematics majors because there are so many other requirements to meet if you want to go to graduate school or an engineering school. I asked my key question and again I got the answer that the differences are not that big. Most of the teachers themselves come from research

universities, and the tenure track system³⁶ acts as a brake on experimentation with content and teaching methods. For instance, in student centered teaching, there is a tension between letting the students be wrong most of the time, and showing them the correct procedures. One difference is perhaps that there is more focus on communication.

In the afternoon, I again asked my key question about what characterizes mathematics education at liberal arts colleges. I got somewhat more specific answers. I talked to Fernando Guovêa. One difference, as compared to other schools, he said, is the Tellos, the goal. Education at Liberal Arts colleges is not vocational and mathematics teaching is not primarily to prepare for graduate school (PhD studies). In particular in upper level courses there is more freedom to choose topics. And even in basic Calculus classes, even if several parallel sections are taught by different professors, they can all have different books and don't have the same exams.

Then I talked to Leo Lifshits. He was also very specific. He said that he always uses classroom time for teaching ideas, not techniques. For instance, in Calculus, he uses a standard textbook³⁷ which has exercises on-line that are automatically corrected. So the students do these as homework. This frees up time for talking about the ideas underlying calculus. I asked if this really works with weak students, and he admitted that there is a self-selection. Students that cannot work like this, or do not want to work like this, take other calculus classes. Still it is an interesting way to work. This way of teaching forces students to think as opposed to doing routine manipulations. That forces a "crisis". Those who cannot deal with it leaves.

But perhaps it can be done in a milder way. In a humanistic way. I'm sympathetic with the idea of challenging students' prejudices about what constitutes a mathematics class and what mathematics is about. That's really the fundamental reason why I'm off on this quest for the perfect mathematics class.

In the evening I had dinner with Jan Holly, her son, and Scott Lambert at a downtown café. It was a nice place, although downtown Waterville had looked a bit deserted when I arrived the day before. I now learned that the college had moved to its present location up on a hill, quite far from downtown, in the early twentieth century, and that the town really didn't identify with the college.

A conclusion of the many conversations I had at Colby College, of which I have reviewed but a few here, is that the personality, knowledge and interests of the individual teacher is essential. What the Liberal arts milieu provides is the freedom to express this in the courses, even though the tenure track system may hold back younger teachers.

5.8 Bates

From Waterville there is just a short drive down to Lewiston. I arrived early and had planned for a stroll downtown and some reading at a nice coffehouse. But it was not to be. The downtown area looked deserted and it was chilly and cold, so I gave it up. But I did find a nice restaurant just by the motel.

At Bates I had a - all too short - conversation with Bonnie Shulman that clarified an issue that I had mixed up in my thinking. It has to do with the

³⁶Professors are provisionally hired for five years during which time their teaching and research is evaluated. If successful, they get tenure.

³⁷Stewart.

relationship between Humanistic Mathematics and Liberal Arts Mathematics - the very core of my project.

It was when Paul Campbell at Beloit College sent me the link to the Journal of Humanistic Mathematics that I became aware of that "sub-culture", as Bonnie said.

At the start of our conversation, Bonnie made it clear that if my key question is how the Liberal Arts context influences how mathematics is taught, then it is not humanistic mathematics that I'm interested in. This drastic way of putting it made it clear to me that Liberal Arts mathematics and Humanistic mathematics are actually different things. I wasn't surprised by the fact itself, but I was surprised by myself having had it mixed up for so long. I said that, well, then what I'm really interested in is indeed humanistic mathematics. But I had thought that I could find it practiced at Liberal Arts colleges.

And of course you can. Nothing prevents a professor to teach humanistic mathematics or teach mathematics humanistically, and it is sometimes done but perhaps not that often. The opportunity is there, but perhaps too often it is a missed opportunity. So in this way, my conversation with Bonnie verified the impression I've got from other conversations, that the Liberal Arts environment may influence mathematics teaching but need not do it.

Another thing that became apparent to me was that what you get at a Liberal Arts college is not so much the classes themselves being taught differently, but the context itself with its focus on breadth, depth, interdisciplinarity, critical thinking, writing and communication et cetera rather than a focus on a vocation or profession.

The American "movement" for Humanistic Mathematics

It would be very interesting to dig deeper into humanistic mathematics itself. What is it, really? I had planned to write about it under this heading but I now realize that I don't know enough, and the text is already long as it is. Perhaps the best I can do is to cite from the *About Page* of the Journal of Humanistic Mathematics³⁸

"The term humanistic mathematics could include a broad range of topics; for our purposes it means "the human face of mathematics." Thus our emphasis is on the aesthetic, cultural, historical, literary, pedagogical, philosophical, psychological, and sociological aspects as we look at mathematics as a human endeavor. More broadly, we aim to provide a forum for both academic and informal discussions about matters mathematical.

The Journal of Humanistic Mathematics was inspired by the work of Alvin White, a former professor of mathematics at Harvey Mudd College. Dr. White was the founding editor of the Humanistic Mathematics Network Journal (HMNJ), a work of love that he almost single-handedly edited and produced for 15 years. Dr. White believed wholeheartedly in the importance of recognizing mathematics as a humanistic discipline and played a significant role in bringing

³⁸<http://scholarship.claremont.edu/jhm/>

this idea to the forefront of many minds. Though this is an independent enterprise, the Journal of Humanistic Mathematics will be building on the spirit and tradition of the HMNJ.”

Let me end by directing the reader to an article about humanistic mathematics education sent to me by Paul Campbell [42].

5.9 Amherst and Wellesley

As I drove down to Amherst in Massachusetts I began to feel that the trip was nearing its end. I decided to have a look at the Atlantic Ocean on the way and did so at Fortunes Rocks near Kennebunkport. Then I headed inland and arrived in Amherst in the early afternoon. After checking in to the College guest house I strolled downtown and had coffee at the Starbucks. It was a Sunday. This was a real college town with a main street with coffee houses, restaurants and bookshops.

The next day I met with Robert Benedetto who was my contact at the mathematics department. Rob had been very helpful during our mail conversation, providing useful information about New England and Amherst, when I planned the trip. Now we had a short conversation before I sat in on one of his calculus classes. What I learned from Rob corroborated what I had learned throughout the trip: there is a certain self-selection of students to liberal arts colleges and the environment encourages more student participation than may be common at other schools. But there is not so great a difference in the teaching although there may be more interaction with faculty from other departments. Liberal arts as such are not explicitly discussed with the students.

While listening to Rob’s calculus class, it dawned on me that what I’ve been seeing during all these sitting-ins at classes, is mathematicians teaching mathematics.³⁹ This is of course obvious, but perhaps the sentence ”it flows so easily from the pen” conveys the feeling I got.

After lunch I talked to David Cox who told me about how the calculus classes are organized at Amherst. Apart from the standard sections running for one term, there is one section that is stretched out over two terms and another section that is more intense, running for less than a term. I thought this was a simple example of a practical humanistic approach to mathematics where the classes are adapted to the needs of the students.

I had planned to continue to Wellesley after a few days, but it turned out to be almost impossible to find any reasonably priced place to stay there. So I decided to stay in Amherst and just drive to Wellesley for a one day visit to the college. This I did on a Wednesday, starting out early before dawn and arriving in Wellesley at around 8 am where I met with Stanley Chang for breakfast. It was Sheldon Rothblatt who had directed me to Stanley.

Wellesley is a womens college and among its many prominent former students we find two US Secretaries of State: Madeleine Albright and Hillary Clinton. Over breakfast I asked Stanley what difference it makes to work at a womens

³⁹It was about the definitions of extreme values, which as Rob stressed, has nothing to do with limits or derivatives. Calculus proper enters with the extreme value theorem for continuous functions on a closed interval. Then ”Fermat’s theorem” was treated leading to a definition of critical points.

college, and during my one-day stay at the college I thought I could detect the amiable atmosphere he had talked about.

As I drove back to Amherst in the evening, I felt that there was now no point in talking to more people - I had already learned more than I could possibly have hoped to learn. The purpose and aim of the college trip was fulfilled. My conversations with Stanley, the classes I sat in on, the lunchtime conversation with several of the faculty and in particular with Alexander Diesl over coffee at the Peet's coffeehouse, it all rounded off the experience.

In the end I stayed for eight days in Amherst and I fell into a daily routine of early morning writing, then breakfast at the Black Sheep Cafe, a stroll on main street, then more work. In the afternoons I did some shopping out at the mall or some leisurely walking. One day I drove a few miles west and had a look at the Connecticut River. I was invited to Sheldon at his Amherst home, and we talked, among other things, about my project and how it had turned out.

As I would be teaching as soon as I got home (after a week of holiday in Paris) I had preparation work to do. And I began to get a little home-sick, or perhaps more like family-sick. For the last two days of the trip I found a place in Natick. I took the train to Boston one of the days. Had I done my home-work I would have known about the JFK presidential library. That has to wait for next time.

6 Mathematical Language

Here I will try to discuss mathematics as a language. In several places I will draw analogies with computer science, in particular the theory of programming languages, which has some similarities with mathematics in that it relies on formalized languages and their relationships with reality.

Magic

One of my colleagues⁴⁰ said something interesting at a meeting at our institute a while ago. It was an observation he had made. It was as if some students, or even many students, when they enter the mathematics classroom they shed⁴¹ the natural logical and rational thinking they use in everyday life and in other academic subjects of study. Instead it is as if magic could solve the problems. We had been talking along these lines a few times before, but now he expressed it explicitly. The point got across and connected to my own thinking. I realized that I've seen the same phenomena. It connected to my own thinking about mathematics as a language and what I have started to name the *The Language Teaching Metaphor*. Let me back up a bit and try to explain my thoughts. It is certainly not a new observation that mathematics can be looked upon as a language. I don't think anyone would deny that. But I think there is much more to it than is normally surmised. One aspect is the invisibility versus visibility paradox referred to above in section 2.

⁴⁰Magnus Lundin.

⁴¹Not being a native English speaker myself, I now and then look up words in a dictionary: "shed" = "to rid oneself of temporarily or permanently as superfluous or unwanted" according to Merriam-Webster's 11:th collegiate dictionary.

The invisibility paradox

It is obvious that our high-tech society couldn't exist without advanced mathematics. It's not just electronics - which relies on physics understood in terms of mathematical physics - it is also the logistics of administrating energy, materials and information that require mathematics and computation. This is all very well known and acknowledged. A thorough and still up-to-date discussion can be found in Lynn Steen's 1985 article *Mathematics: Our invisible Culture* [2]. Mathematics is central to our technology, society and culture, yet goes unnoticed most of the time. The applications are invisible, mathematics is built into our society and technology and it is only visible to the engineers, economists and scientists (mathematicians included) that work on a day-to-day basis with maintaining and developing it. Even many students in engineering never use much mathematics in their jobs after graduating. Mathematics may be visible while studying at school and university but not after that. Indeed, for almost everyone, mathematics is highly visible in school but invisible outside school⁴².

One way of understanding the visibility of school mathematics is precisely by analogy with natural language, that is: any spoken and written human language. When you speak your own native language, then you are not, most of the time, conscious of that. It's just something you do. You're probably more conscious when writing since that is more difficult - it is not so instantaneous - and it is more reflective.⁴³ However, when you speak a foreign language you are much more conscious of speaking.

Now, could it be that when mathematics teachers use the mathematical language⁴⁴ they are not aware of the fact that they are using it? They speak, and write, as if the students were as fluent as themselves? It's like in Foreignland long time ago when people couldn't understand that not everyone spoke Foreignish. But the typical student is not fluent. Mathematics is not a native language for most people.

Formal mathematics as taught in school is therefore highly visible to most people. But the mathematics that is built into our society and technology is almost entirely invisible. Of course it is not formulas that are built into technology. There are no formulas in a cell phone. But mathematics was needed when designing it and the network infrastructure that makes it work. And all that is based on our knowledge of physics described in mathematical language. But can I really understand how mathematics is built into a cell-phone? I mean, in some detail: could I give a plausible explanation to a student asking me?

We use embedded mathematics all the time without being aware of it but not formal mathematics which is used by almost no-one except mathematicians and a subset of scientists, engineers, economists and logisticians and the like.

Plato and Object Oriented Programming

This is very strange. It is as if mathematics is built into the very fabric of reality. Most languages (as far as I know) have nouns, adjectives and verbs. This corresponds to the fact that the world consists of "things" that have "properties"

⁴²This visibility paradox is one background to the "Is Algebra Necessary" discussion, see section 5.3.

⁴³The reflective nature of writing is indeed one point of writing.

⁴⁴I'm thinking of both the formalism itself and the natural meta-language needed to communicate mathematics

and can "perform actions". This also, by the way, corresponds closely to the classes of Object-Oriented-Programming (OOP): classes are blueprints of things (abstract or concrete) that have properties and actions. It is a close step to think of classes as Platos ideas and the objects (the instantiations of classes, still speaking OOP) as concrete physical things.

However, this view of reality and language can be questioned. Bonnie Schulman at Bates College gave me an article she had written, which among other topics, discusses the noun - verb - adjective view of the world [43]. I haven't had time to think more deeply about this. To better understand the mappings between reality and our descriptions of it and the categories and language constructs we use, must be important for mathematics teaching.⁴⁵

The Symbols of Mathematics

Suppose provisionally that mathematics is about ideas, ideas somehow connected to phenomena in reality. These ideas must be captured by some kind of formalism using some kind of symbols. The ideas of mathematics may start out as vague, but eventually they have to be made precise, or exact, because we want to prove beyond any reasonable doubt that our theorems, phrased in terms of these ideas, are correct, giving the correctness of the underlying axioms or foundations.⁴⁶

I have the got the impression that most teachers, if they think about it at all, consider the formalism itself and the symbols themselves, to be very exact. I don't think so. More and more I have come to think about the symbols and formalism of mathematics as being inherently vague. How can a symbol, however elaborate and decorated by pre-fixes, suffixes, indices and what-not, capture the full body of a complex mathematical concept? Here's an example from topology.

Let K be a finite simplicial complex. The n -th homology group is denoted by $H_n(K)$ and defined by

$$H_n(K) = Z_n(K)/B_n(K)$$

Even if, or perhaps in particular if, you don't know what a *finite simplicial complex* is or what a *homology group* is, it should be clear that they are complicated objects all of whose properties cannot possibly be captured by the symbols K and $H_n(K)$. First, K is just a *name* for the finite simplicial complex under consideration. Secondly, the symbol $H_n(K)$ is a little bit more elaborate, but in another context it could stand for something entirely different, a function for instance. Explaining what $Z_n(K)$ and $B_n(K)$ stand for helps a little, but then the reader must rethink the meaning of $/$. There is a lot of conceptual understanding, based on many concrete examples, that is denoted by this piece of mathematical formalism.

Context is the key word here. Mathematical symbols and even whole formalisms, say nothing without a context. The symbols have connotations given

⁴⁵This leads over to philosophy of mathematics, and questions about the nature of mathematical objects, see section 7.

⁴⁶It must be realized that basing mathematics on secure foundations is an "after-the-action" reconstruction of mathematics, it is not in general how new mathematics is discovered. See for instance [44] and [43].

by that context. In order to even read a formula, parse it so to speak, and understand it, you must have some picture of the context. When you are learning new mathematics, part of the problem is that you don't have that context yet.⁴⁷

My contention is that the symbols and formalism of mathematics cannot in principle capture all aspects of a mathematical concept. Therefore the formalism is inherently vague to a considerable extent. To make things worse, symbols are often used in slightly different ways in different contexts.

An analogue with computer science is useful here too. In order for a computer (a program really) to work with a concept or an idea, it must be completely captured by the "symbol" in this case the appropriate data structure or class. Everything must be encoded. The program may have "background knowledge" but has no imagination and it cannot use any information not programmed into it or learned in some way. This is drastically different from how we as human beings work. After all, we are not machines.

The Language Teaching Metaphor

These observations bring us over to what I have started to name the *The Language Teaching Metaphor*. There is an analogy with computer science and there is an analogy with teaching natural language. Computer scientists, in particular when discussing the theory of programming languages, speak of *syntax, semantics and pragmatics*. Syntax is precisely the syntax of the language, its grammar, its rules for how the words and phrases of the language can be put together without committing any errors. A program must be syntactically correct, otherwise the computer cannot run it. Likewise, a piece of written mathematics must be syntactically correct in order to make sense. The same holds for sentences and text written in a natural language. But as we proceed from computer programs to written mathematics to written natural language, we can tolerate a few errors in mathematics, and perhaps quite a few errors in natural language. We understand anyway.⁴⁸ The computer tolerates no syntactical errors at all.

With semantics we focus on the meaning of what is written. In a program this is what the program is supposed to do when it runs. A program can run but it may not do the right thing, it may not do what it was intended to do. The same goes for a piece of written mathematics. You may require a function to be zero in order to find points where there could be a maximum or a minimum. You may solve the equation correctly. Everything is syntactically correct. But of course, the semantics is all wrong. To find candidate points for extrema you should start by differentiating the function first.⁴⁹ Likewise, the semantics of written text is the very motivation of writing in the first place - to convey a message - to communicate.

Pragmatics have to do with how the language is used in practice. Different programming languages are used for different programming tasks, partly out of tradition, but more so since they are constructed with different applications in

⁴⁷In bad mathematics teaching the problem is aggravated by the lack of *narrative*, something that is often the case with mathematics textbooks. Providing the narrative is humanism. That narrative is central to mathematics teaching soon becomes clear when you sit in on good college mathematics classes, as I did.

⁴⁸Although reading texts with lots of grammatical errors throws uncertainty on the meaning and it is often quite exhausting.

⁴⁹On the last calculus exam I gave, a student complained that he couldn't solve the equation $f(x) = 0$. It was an extremum problem.

mind. The same goes for mathematics, methods are chosen that are thought to be appropriate to a given problem and symbols are chosen in accordance to that. Let me take an example of a pragmatic question in calculus. Let's say we are teaching calculus. What is the basic pragmatic question when doing derivatives? I would say it is

What kinds of problems are the derivative the answer to? And how to use it in such cases?

If the topic is integrals, the corresponding questions are

What kinds of problems are the integral the answer to? And how to use it in such cases?

The first parts of the questions "What kinds of problems ..." are humanistic questions. They are about concepts and classical problems, history, philosophy and culture. The second parts of the questions "And how to use it ..." are more about skills, they are practical and applied.

I think one problem with much of traditional mathematics teaching is that it is mostly concerned with the syntax. Symbols are manipulated according to the rules, now and then with backdrop to semantics, but seldom in any pragmatic humanistic context.

But how is all this related to natural language and the teaching of natural language? I will take French as an example. A good course in French must consist of three things

- The grammar of French
- The literature, culture and history of France
- How to use French in practice in various circumstances such as reading, speaking and writing.

It may not be entirely one-to-one, but I see a clear correspondence to

- Syntax
- Semantics
- Pragmatics

I don't think French teachers think of their subject or their teaching as particularly *logical* or *rational*. These are not concepts that naturally attach to natural language teaching - apart from grammar.⁵⁰ But I do think their teaching is highly logical and rational. And I don't think the students shed their logical and rational thinking and their common sense when they enter the French classroom. Studying language is a rational and humanistic endeavor.

How come then that mathematics, a subject that ought to be the most logical and rational of all human studies, is attempted by first leaving common sense outside the door and relying on pure magic? I think the reason is that mathematics has ever since first grade been disconnected from language. Mathematics is conceived as something entirely different from language. Learning to

⁵⁰I remember being extremely annoyed when the German teacher in school said that German grammar was as logical as mathematics. I can accept that today.

read-and-write and learning to count are conceived of as totally disconnected activities.

It is said that mathematics is the language of nature. Well, aren't all human languages, languages of nature and culture? Why else would we need them? The reader may now be curious: how would you teach mathematics as a language in this way? I will return to that question elsewhere [45].

Mathematics writing

I came across a hilariously funny book by Morris Kline [14]. It's title is *Why the Professor Can't Teach* and it laments the poor quality of mathematics teaching and tries to explain why it was so - "was" - because surely the situation must have improved by now. It was published in 1977. In a chapter about mathematics texts, Kline writes

"Many authors seem to believe that symbols express ideas that words cannot. But the symbolism is invented by human beings to express their thoughts. The symbols cannot transcend the thoughts. Hence, the thoughts should first be stated and then the symbolic version might be introduced where symbols are really expeditious. Instead, one finds masses of symbols and little verbal expression of the underlying thought."

This paragraph confirms what I have written above about the formalism and symbols of mathematics. Kline also writes

"Student interviews, discussions, and dialog quickly revealed that what the student sees when looking at a graph is not what the teacher sees. What students hear is not what the instructor thinks they hear. Almost nothing can be taken for granted. Students must be taught to read and interpret the text, a graph, an expression, a function definition, a function application. They must be taught to be sensitive to context, to the order of operations, to implicit parentheses, to ambiguities in mathematical notation, and to differences between mathematical vocabulary and English vocabulary when the same words are used in both. Interviews revealed that the frequent use of pronouns often masks an ignorance of, or even an indifference to, the nouns to which they refer. The weaker student has learned from his past experience, that an instructor will figure out what 'it' refers to and assume he means the same thing."

This is also my experience from many years of teaching and contemplating what's happening when teaching. In the preface to the book I'm writing with a colleague⁵¹, we write "The only thing a teacher speaking in front of an audience can be sure of is that everyone thinks about something different than the speaker." And we continue to say that this is why good writing is so important in mathematics. The spoken word is transitory - it briefly passes through the lecture hall or the class room - while the text remains and can be read again and again. That is, in case there is something to read on the pages.

⁵¹Konsten att räkna, Anders Bengtsson and Mats Desaix.

Kline laments, as many other authors do, the poor writing in mathematics textbooks. And even if the writing is not poor, it is brief. A mathematics textbook often consists mainly of formulas, figures, examples and exercises with just short segments of explanatory text in between. This text is submerged in all the rest. It does not stand out, and since it is often tersely written, it is not easy to understand. The connection between the text and the formulas is weak. It is as if the formulas, in the imagination of the author, in some magical way speak for themselves - which they don't.⁵²

This is not to say that there aren't any well written mathematics books. There are many popular books about mathematics that does a good job. There are also quite a few books written for liberal arts mathematics courses, for instance [46, 47, 48]. And an Internet search brings up many more as well as links to liberal arts mathematics courses. What is lacking - I believe⁵³ - are well written and readable textbooks for the standard university courses in algebra, linear algebra and calculus.

There is also a problem with the expectations of the students. They are used to mathematics books where you skip the text entirely, often also the formulas, and they go directly to the examples and the exercises.⁵⁴ Why don't we assign a pack of books - as is common in social science and the humanities - at least a textbook, a popular book and a liberal arts book? Doing that, we would have to teach that way and examine the course that way too.

Let me end this section by mentioning a quip I've read and heard a few times lately - the fear that school mathematics may go the way of Latin - that is: disappear. I just read one take on this in a New York Times article [49] that preceded the Hacker article, but discussed the same issues, arguing for applied mathematics in schools

"Traditionalists will object that the standard curriculum [algebra - my insertion] teaches valuable abstract reasoning, even if the specific skills acquired are not immediately useful in later life. A generation ago, traditionalists were also arguing that studying Latin, though it had no practical application, helped students develop unique linguistic skills. We believe that studying applied math, like learning living languages, provides both usable knowledge and abstract skills."

Indeed - study mathematics as a **living language** with a culture - like French!

7 Mathematical Reality

What is mathematics about? Is there a mathematical reality, or a reality to mathematical concepts, and in that case, where is that reality located? Such questions, I presume, are seldom discussed in the classroom. I think they should. Integrating them into courses is one of my ideas about how to humanize mathematics education. But wouldn't that be a complete waste of time? Not if it

⁵²One of the references I've lost is to a recent Swedish PhD dissertation in pedagogy where it is studied (among other things) how students jump between the formulas in mathematics books, not reading the text.

⁵³By writing this I may be proven wrong.

⁵⁴There is also an art to reading mathematics texts. At Colby College I got a reference to Simonson's and Gouvêa's *How to Read Mathematics*:
<http://www.people.vcu.edu/~dcranston/490/handouts/math-read.html>

opens new roads into an understanding of the esoteric formalism and language that mathematics use to capture this very reality, whether it exists or not.

Isn't it very strange that a subject that deals with abstract objects seldom discusses with its students what these objects are, or where they are? Is it a wonder that most people have problems with mathematics? Who wouldn't have problems understanding a subject that is about things you don't know what they are or where they are?

There is an old quip from Bertrand Russel:

"Mathematics may be defined as the subject where we never know what we are talking about, nor whether what we are saying is true."

This was written in 1901 [50] in the context of Russel's grand attempt at reducing all of mathematics to logic [51]. Perhaps it also captures, unintentionally, how many students come to view mathematics.

One common objection to this line of argument is: students don't want it, they only want to know how to solve the problems. But how would they know what to want if they've been taught mathematics for years as trying to solve problems that make little sense except as routine exercises? It is we as teachers who give them the problems to solve, and we can give them other problems. No, I don't buy that objection.

Sometimes answers to questions about mathematical reality collapse down to a duality: Either mathematics exists "out there" and is discovered by the mind, or it's all a mental construction, and is consequently invented by the mind. However there are many nuances and the question has a similarity to the old philosophical question from the middle ages about the existence of universal concepts. I got this notion from Lars Mouwitz's PhD dissertation [4]. Four distinct directions of thought crystallized (in my interpretation).

Universalialia ante res Universal concepts come first. This is the concept realism of Plato: concepts are not invented, they exist before reality and real things are copies of the concepts. Mathematical concepts are real (in this sense) and are discovered by the mind. This is Mathematical Platonism. It is sometimes jokingly said that most mathematicians are Platonists on weekdays due to the very strong feeling they have that they are working with real existing objects. A critique against this view is based on the obvious problem of locating where the concepts actually reside.

Universalialia in res Universal concepts reside in real "things". This is Aristotle: concepts exist in reality and are extracted by the mind - not invented, but discovered. They are built into things. Knowledge is empirical and abstracted. Mathematics becomes the language of nature.

Universalialia post res Universal concepts come after real "things". This sounds like a more modern view. There are no concepts where there are no minds. Concepts are invented by the mind based on empirical studies, but they don't exist in physical reality itself. Concepts are cultural phenomena, propagating through society (space) and history (time). An analogy would be the "memes" of Dawkins. In pedagogy and philosophy, this is constructivism. Concepts are created by the mind in the mind. A question is: how

can concepts be private, yet commonly shared, correct and useful? An answer could be: by social and cultural processes and communication, and the concept-forming minds are parts of reality. Mathematical concepts are social constructions describing phenomena in reality.

Nominalism There are no concepts, only the things themselves. Concepts are just names. In mathematics this corresponds to formalism. Still jokingly, when the weekday Platonism of mathematicians is challenged, they resort to Sunday formalism. Mathematics is about nothing, it is just a play with symbols. Wittgenstein held the view that there is nothing beyond the signs and symbols, it's all language-games. Perhaps many students end up here. The mathematical formalism doesn't mean anything, the symbols are disconnected from reality. It becomes a meaningless and largely incomprehensible game.

To me, the first and last positions are too extreme. Platonism may be a beautiful idea, but is it scientifically plausible? Nominalism is too poor. A problem with full constructivism (mathematical concepts as social constructions) is the Wigner problem: how can it be that a human construction such as mathematics so closely models the behavior of physical reality? [52]. There is no denying - I think - that mathematical principles seem to be built into physical reality. So some kind of compromise between the second and third viewpoint may be the most viable.

In his book *What is mathematics, really?* [53] David Hersh argues for mathematics being a social construction, a part of human culture. If mathematics is a shared social construction, then that could account for the feeling that mathematics exist "out there somewhere" external to the individual mind and is somehow discovered. Yet it is invented by the minds.

This is an idea with precursors. There is an interesting article by Leslie White in *The World of Mathematics* [54], that forcibly argues that mathematics is a cultural phenomenon. White, who was an American anthropologist, addresses the question of where the mathematical concepts and truths reside. Do they belong to the external physical world or are they human mental constructions? His text is interesting in many ways. He refers back to earlier discussions about this issue and he is very eloquent about his own point of view: mathematics is purely a cultural phenomenon.

Much of what White writes make perfect sense, but he carries the argument too far. In reducing all of mathematics to a purely cultural phenomenon, completely ruling out the role of the individual mathematician, and physical reality, his concept of culture takes on a metaphysical character in itself.

Another way to phrase the question is to ask whether mathematics is discovered or invented. White reviews how mathematicians have the feeling that they are discovering something which is external to themselves, citing G.H. Hardy as an example (among others). On the other hand, it seems just as clear that mathematical concepts are human inventions. So the answer would be that mathematics is both discovered and invented. White clearly renounces any notion of a Platonic abstract realm where mathematics reside.

White's answer to the dichotomy of discovery versus invention is to claim that mathematics is a cultural phenomena. This sounds reasonable, but he does it in an anthropological framework which to me takes it too far. When thinking

of mathematics as a mental phenomenon there are of course two senses to the concept of a mental: mental concepts in the individual human being and shared mental concepts of the species. Here are some quotes from White (where the italics are mine).

”What we propose to do is to present the phenomenon of mathematical behavior in such a way as to make clear, on the one hand, why the belief in the independent existence of mathematical truths has seemed so plausible and convincing for so many centuries, and, on the other, to show that all of mathematics is nothing more than a particular kind of primate behavior.”

Clearly it is too simple-minded to consider ”external physical reality” and ”internal mental reality” as the only possibilities for where mathematics could reside. The human culture is another possibility. White writes

”Mathematical truths exist in the cultural tradition into which the individual is born, and so enter his mind from the outside. But apart from cultural tradition, mathematical concepts have neither existence nor meaning, and of course, cultural tradition has no existence apart from the human species. Mathematical realities thus have an existence independent of the individual mind, but are wholly dependent upon the mind of the species.”

This makes sense to me, but then he continues with

”Or, to put the matter in anthropological terminology: mathematics in its entirety, its ’truths’ and its ’realities’, is a part of human culture, nothing more.”

It’s the nothing more part I don’t agree with. These quotes are from the beginning of the text. White then goes on to argue his case and much of it makes sense, but as said, he takes the argument too far. Even physical theories like Maxwell Electrodynamics and Einstein General Relativity become purely cultural in White’s view. Certainly, the detailed formulation of the theories of fundamental physics are dependent upon culture, but there is a basis in physical reality, independent of human culture, or any culture anywhere. But to White, culture is a ”super-human” entity that could only be explained in terms of itself. The role of the individual is reduced to null: discoveries are not made by individuals, they are something that happen to them. In this way, culture becomes meta-physical.

In more recent times, Reuben Hersh has argued for mathematics being a cultural phenomena. The argument is popularized in his book *What is Mathematics, really?*. Another reference is his article *Some Proposals for Reviving the Philosophy of Mathematics* in [55]. Hersh does not take the argument as far as White.

The arguments for mathematical concepts residing in our common human culture are very convincing, but do they explain everything? Still we have the Wigner problem of the *The unreasonable effectiveness of mathematics in the natural sciences* [52]. I cannot escape the feeling that there must be something in physical reality that serves as a basis for mathematics. A point of view very different from the ”cultural basis view” is put forward by Roger Penrose in his

The Road to Reality [56]. Penrose is a Mathematical Platonist and his discussion on the interactions between three worlds: the Platonic mathematical world, the Physical world and the Mental world in the first chapter is very intriguing. Where is culture in that picture?

These questions are also reminiscent of Gottlob Frege's struggle in *The Foundations of Arithmetic* [57] with defining numbers independently of individual mental states in order that mathematics not become a part of psychology [58]. A compromise view would be that mathematics is both discovered and invented: the concepts are invented whereas the truths are discovered.

Discussions like these are closely related to questions about the foundations of mathematics and debates more than a hundred years ago. The history of the major philosophical strands of logicism, formalism and intuitionism is nicely summarized in David Hersh's article *Some Proposals for Reviving the Philosophy of Mathematics* in [44]. Another reference is Morris Kline's *Mathematics, The Loss of Certainty* [59], a book that would work very well in a Liberal Arts-inspired mathematics course.

Discoveries in analysis such as, for instance, continuous but nowhere differentiable functions and Fourier series showed that the geometric intuition underlying infinitesimal calculus was insufficient. This led to arithmetization of analysis and Cantor's set theory. Then to the Russell paradox and the breakdown of Frege's system of logic and mathematics. The classical foundational programs of logicism, formalism and intuitionism were all attempts to resurrect certainty of mathematical knowledge. They all failed. Hersh writes that mathematics has no foundations and needs no foundations. This is a point of view that I think is quite controversial.

Certainty of knowledge has been a preoccupation of philosophers of all time, in particular since Descartes and onwards. Today, this preoccupation seems antiquated. Of course, our scientific knowledge of for instance fundamental physical reality is certain to a very high degree, but no-one claims it to be 100% certain or even hope for it ever to be. The modern scientist can live with uncertainty. Indeed, if you can't stand living with uncertainty, then you're no scientist at heart. Mathematics is very likely to be even more certain than fundamental physics. But isn't it more interesting if there is an epsilon risk of error rather than zero risk? And historically, no paradox or inconsistency ever discovered has been able to destroy mathematics. The only consequence of the Russell paradox is: don't deal with such silly ideas. Isn't it obvious from the very beginning that the idea of the set of all sets is ill conceived?

Anyway, my real question is not what is the best philosophy of mathematics, but instead: these kinds of discussions, may they help in teaching and learning mathematics? In my view, it could infuse a subject that for many students seems very dry with life.

8 Concluding remarks

To conclude, let me return to where I started, with the invisibility of mathematics as described in Lynn Steen's article [2]. The text is written in 1985 and has a distinctly American perspective, but I think its contents are relevant in Sweden and still today. The main argument is that most of modern mathematics and

its applications in technology and society are largely unknown to the general public, it is invisible, forming an invisible culture. The known mathematics is old mathematics. In one drastic phrase, Steen writes

”In contrast [to the situation regarding science and technology], public vocabulary concerning mathematics is quite primitive: it is not a decade, not a century, but a millennium out of date.”⁵⁵

The third time I read the article, I had had the discussion with one of my colleagues referred to above about mathematics perceived as magic. It then struck me that Steen uses analogous imagery when describing the status of mathematics education. It is *wizardry*. One comment I sometimes get when discussing alternatives to today’s mathematics teaching is that students don’t want that, they are not interested in why it works, they just want to know how to use it. I’m not saying that’s not generally true. I’m saying that we need not as teachers succumb to such a naive view. I find support in Steen writing

”Yet, to be honest, this is the only level [cultural literacy as opposed to practical and civic literacy⁵⁶] on which the arcane research of twentieth century mathematics can truly be appreciated - as an invaluable and profound contribution to the heritage of human culture.”

A humanistic, liberal arts inspired approach to mathematics may indeed be what is needed at the university level.

The paradoxical invisibility of mathematics in society, discussed above in section 2, leads to the awkward question of: why teach advanced mathematics at all? Except for the relatively few who study to become engineers or scientists or mathematicians. To the vast majority of people, advanced mathematics is superfluous.

I’ve lived with this question since I started to teach at the University of Borås. I don’t think I thought about it before that. I remember one colleague saying that he never used the mathematics he learned at engineering school. I have other colleagues claiming that what little mathematics are needed in their applied courses can be picked up there, implying that the mathematics courses are really un-necessary. I don’t know how common sentiments like these are. This is certainly anecdotal evidence.

But there must be something wrong here. My institute has just recently gone through an evaluation process, where as a part, descriptions of the educational programs have been written. To me it seems that there is quite a lot of mathematics in at least some of the applied courses. I also had the experience about a year ago of working through a course in polymer chemistry to make the mathematics more comprehensible. It was clear that there was mathematics throughout the course. One example being differential equations for reactions. Another example was partial integration to calculate the average length of polymer chains.

⁵⁵The text continues with ”Explaining what is actually happening in contemporary mathematical science to the average layman is like explaining artificial satellites to a citizen of the Roman Empire who believed that the earth was flat.

⁵⁶Here Steen is referring to Benjamin Shen’s three aspects of literacy in science. I haven’t been able to find a reference.

Can it be that teachers in applied courses don't recognize the mathematics of their own education in the applied courses they teach? Is the mathematics of the applied courses so strongly tied to the application that we have a kind of backwards transfer problem: the teachers themselves don't understand where their knowledge comes from?

And this leads to the need for more communication between teachers in applied courses and the mathematics teachers: it's all about language, reality and communication.

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Carleton College Classes: *Calculus with Problem Solving* with Deanna Haunsperger, *Real Analysis* with Gail Nelson, *Combinatorics* with Eric Egge, *Calculus* with Sam Patterson.

Conversations: Deanna Haunsperger, Stephen Kennedy, Andrew Gainer-Dewar and Brian Shea.

Macalester College Classes: *Number Theory* with David Bressoud, *Applied Calculus* with Chad Topaz, *Statistical Modelling* with Daniel Kaplan, *Math and Politics* with Karen Saxe and *Combinatorial Games* with Andrew Beverage.

Conversations: Karen Saxe and David Bressoud.

St. Olaf College Conversations: Paul Zorn and Ted Vessey.

Oberlin College Classes: *Vector Calculus* with Susan Colley, *Topics in Contemporary Mathematics* with Michael Raney, *Foundations of Analysis* with Michael Henle, *Calculus* with Michael Raney.

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